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EXACT SOLUTIONS AND NUMERICAL ANALYSIS OF THE PROBLEM OF AN INTENSE EXPLOSION IN CERTAIN IDEAL COMPRESSIBLE MEDIA*

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The well-known selfsimilar problem of an intense explosion in an ideal compressible medium possessing a certain arbitrariness in the form of the internal energy is considered. The problem was formulated by Sedov. The existence of the first two integrals reduces the problem to the study of the integrability of a single, first-order differential equation.

We will show that even in the simplest case when the problem has planar symmetry, and the equation reduces, in the general case, to an Abel equation with functional coefficients which is not integrable in quadratures. A special case of its integrability is found, which enables us to write out the analytic solutions of the problem for a certain family of media including real and dust-containing gases (under the assumption that the phase parameters are in equilibrium). The results generalize the results obtained earlier /1-4/. All solutions obtained can be continued to the plane of symmetry, and their asymptotic behaviour near it is investigated.

A numerical analysis of the problem is carried out for the same family of media for the cylindrical and spherical cases. Two new effects are found for disperse media such as a liquid with bubbles and a dusty gas (previously studied numerically in /5, 6/), namely the non-monotonic form of the velocity behind the shock wave, and the effect of incompressibility when the mixture contains a fairly small amount of gas. In the spherical case the limit solution of the problem, when the amount of gas is reduced, is represented by the well-known solution of the problem of an intense explosion in an incompressible fluid.

1. We shall give a brief formulation of the problem (given in greater detail in /1, 2/). Let the internal energy density have the form

$$e(p,\ \rho) = p\varphi(g)/\rho_0, \quad g = \rho/\rho_0 \tag{1.1}$$

where p and ρ are the pressure and density, and φ is an arbitrary function. In this case the problem is selfsimilar (two independent dimensional constants are the energy of the explosion, and ρ_0 is a constant with dimensions of density), and the dimensionless density g, the velocity $f = v/\dot{x}_s$ and the pressure $h = p/(\rho_0 \dot{x}_s^2)$ satisfy the system of three first-order ordinary differential equations obtained from the equations of continuity, motion and conservation of entropy within the particle:

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$$(\lambda - f)\, g'/g - f' - (\nu - 1)\, f/\lambda = 0, \quad h' - (\lambda - f)\, gf' - {}^1/_2\nu fg = 0 \tag{1.2}$$

$$(\lambda - f)\,(h'/h - g'd \ln \chi/dg) + \nu = 0 \cdot \quad \chi(g) = \varphi^{-1} \exp \int (g^2\varphi)^{-1} dg$$

where $\lambda = x/x_s$ is the dimensionless coordinate (a prime denotes differentiation with respect to λ), the parameter ν is equal to 1, 2, 3 for plane, cylindrical and spherical symmetry respectively, and x_s and x_s' are the coordinate and velocity of the (assumed) intense shock wave. From considerations of dimensions it follows that $x_s \sim t^{2/(\nu+2)}$ and t is the time, and this was utilized in (1.2). The following relations on the intense shock wave $(\lambda = 1)$ serve as the boundary conditions for system (1.2):

$$f_s = 1 - g_1/g_s, \quad h_s = g_1 f_s, \quad 2g_1\varphi(g_s) = 1 - g_1/g_s \tag{1.3}$$

where g_1 is the initial density of the medium and the index s denotes the values of the variables on the wave. The last expression in (1.3) should be regarded as an equation in g_s, (and since its right-hand side should be positive, the condition $0 < \varphi(g_s) < 1/(2g_1)$ should also hold).

In order to make the solution physically meaningful, we must require that the condition for there to be no sources at the centre of symmetry (explosion) must also holds

$$f(0) = 0 \tag{1.4}$$

If we digress from determining the dimensional characteristics of the flow, then the boundary value Problem (1.2)-(1.4) will represent a mathematical formulation of the problem of an intense explosion in a medium with internal energy of the form (1.1).

The existence of two algebraic first integrals of system (1.2), namely of adiabaticity and energy:

$$h\chi^{-1} g\,(\lambda - f)\, \lambda^{\nu-1} = C_a \tag{1.5}$$

$$[hf - g\,(\lambda - f)({}^1/_2 f^2 + h\varphi)]\, \lambda^{\nu-1} = C_e \tag{1.6}$$

where C_a and C_e are found from (1.3); $C_e = 0,\ \forall\varphi$ enable us to simplify the investigation of the problem of an intense explosion, and to pass to the problem of considering the Cauchy problem for a single first-order equation in the variables g and $F = f/\lambda$:

$$d \ln g/dF = g\varphi\,(F - 1)^{-1}\{F\,[(\nu - 1)\,F + 1] + g\varphi\,(F - 1)[(\nu - 1)\,F + \tag{1.7}$$

$$1 - {}^1/_2\nu]\}\,\{g^2\varphi^2\,(F - 1)(F - 1 - {}^1/_2\nu) + g\varphi F\,(F - 1 - \nu) + \tag{1.8}$$

$${}^1/_2\nu F^2\,(1 - g^2 d\varphi/dg)\}^{-1}$$

$$F_s = 1 - g_1/g_s, \quad 2g_1\varphi\,(g_s) = 1 - g_1/g_s$$

Condition (1.4) is verified after obtaining the solution of Problem (1.7), (1.8). Eq. (1.7), which we study with $g\varphi$ as the unknown and with the function $F(g)$ given, represents a Riccati equation and can be integrated in quadratures from $F(g)$ and this, together with the use of first integrals, enables us (see /1/) to describe certain general properties of the behaviour of the solution of the problem with an arbitrary function $\varphi(g)$. The question of a straight-forward integration of (1.7) remains open.

2. In order to study the integrability of Eq. (1.7), we shall reduce it to a more convenient form. Let

$$u = 1/(g\varphi), \quad w = 1/F - 1 \tag{2.1}$$

Then Problem (1.7), (1.8) can be written in the form

$$\{2u\,(w + \nu) - w\,[\nu + (2 - \nu)\,w]\}\, w^{-1}\,(w + 1)^{-1}\, dw = \nu du + \{\nu u^2 - u\,[\nu + \tag{2.2}$$

$$2\,(\nu + 1)\,w] + w\,[\nu + (2 + \nu)\,w]\}\, d \ln g \tag{2.3}$$

$$w_s = u_s/2, \quad u_s \equiv u\,(g_s) = 2g_1/(g_s - g_1)$$

Knowing the particular solutions of Eq. (2.2), we can conclude that the solution of Problem (2.2), (2.3) must be contained within the "beaker" $g \geqslant 0,\ 0 \leqslant w \leqslant u\,(g)$.

Let us now assume that g, w, u are independent variables. Then (2.2) will be a Pfaff equation of not fully integrable type (see e.g. /7/ for details on Pfaff equations). The one-dimensional integral manifolds of this equation are described, in the space g, w, u, by the following system of algebraic equations:

$$(u - w)(w + 1)^{2(\nu-1)/\nu} = w^2\theta\,(g), \quad u - 1 - (1 + 2/\nu)w = -d \ln \theta/d \ln g \tag{2.4}$$

where θ is an arbitrary function of g (we can assume that system (2.4) represents a general "parametric" solution of Eq. (2.2) with parameter $\theta\,(g)$).

In the case of plane symmetry $(\nu = 1)$, system (2.4) takes its simplest form. Eliminating

w from (2.4), we can obtain a differential equation for θ when the function $u(g)$ is given. If on the other hand we take, instead of θ,

$$Y = (1 + 4u\theta)^{-1/2} \tag{2.5}$$

then the variable Y will satisfy the Abel equation

$$dY/d\tau + Y(1 - Y)[(\alpha + \beta)Y + \alpha] = 0 \tag{2.6}$$

$$\tau = \ln g,\ 2\alpha = d\ln u/d\tau + \quad u,\ \beta = 3u \tag{2.7}$$

The formula

$$w = 2uY/(1 + Y) \tag{2.8}$$

obtained from (2.5) and the first relation of (2.4), enables us to reformulate Conditions (2.3) thus:

$$Y_s = 1/3,\quad u(g_s) = 2g_1/(g_s - g_1) \tag{2.9}$$

Two trivial solutions of Eq. (2.6), namely $Y \equiv 0,\ Y \equiv 1$, do not satisfy the first condition of (2.9). If we can find, for the given function $u(g)$ (or $\varphi(g)$, see (2.1)), the solution of the Problem (2.6), (2.9) in the form $g = \ (Y),\ Y \in (0; 1)$, then we can also obtain the following parametric relations connecting λ, h, f with Y:

$$\begin{gathered}\lambda^3 = 2C_a\chi\varphi^2(1 - Y)Y^{-2}(1 + Y)^{-2}[1 + (1 + 2/(g\varphi))Y]^3 \\ h = \lambda^2\varphi^{-1}Y(1 - Y)^{-1}(1 + Y)^2[1 + (1 + 2/(g\varphi))Y]^{-2} \\ f = \lambda(1 + Y)[1 + (1 + 2/(g\varphi))Y]^{-1}\end{gathered} \tag{2.10}$$

from integrals (1.5), (1.6) and formulas $f = \lambda F$ (2.1), (2.8).

3. We can easily find a simple example of the integrability of (2.6). Let

$$\alpha = k\beta,\quad k = \mathrm{const} \tag{3.1}$$

It is clear that in this case the variables in (2.6) are separable. By virtue of (2.7), relation (3.1) (which is a differential Bernoulli equation), defines a set of functions

$$u = (a + bg)^{-2},\ a = 6k + 1,\ b = \mathrm{const} \tag{3.2}$$

for which Eq. (2.6) can be integrated.

The solution of Problem (2.6), (2.9) for the functions (3.2) depends on the constants a, b and g_1, but in the present case we can write $g_1 = 1$ without loss of generality. A solution with a shock wave of the problem of an intense explosion may exist (see (1.3)) only when the conditions

$$g_s > 1,\ u(g) > 0 \tag{3.3}$$

restricting the values of a and b are imposed on the solution. Let us first consider the case of $a = 0$, where we have $u = (bg)^{-1}$ or $\varphi = b$. The problem of an intense explosion in such a medium was studied earlier by numerical methods for $\nu = 3$ /1/. A solution of Problem (2.6), (2.9) of the form

$$\frac{1}{g} = 1 - 2b - \frac{b}{2}\ln\frac{(1 - Y)(2Y)^4}{(5Y - 1)^5} \tag{3.4}$$

with the values of the parameter Y lying within the half-interval $[1/3; 1)$ yields, together with (2.10), an exact parametric solution of the problem of an intense explosion at $\nu = 1$. From Conditions (3.3) it follows that $0 < b < 1/2$. The solution (3.4), (2.10) can be continued to the plane of symmetry $(Y = 1)$, and behaves in its neighbourhood as follows: $\lambda \sim -\sqrt{\varepsilon}\ln\varepsilon$, $g \sim -(\ln\varepsilon)^{-1}$, $h \sim 1$, $f \sim \sqrt{\varepsilon}$ when $\varepsilon \equiv 1 - Y \ll 1$. The extendibility of the solution implies that Condition (1.4) holds.

When discussing the case of $a \neq 0$, it is convenient to introduce new constants, namely $\gamma = 1 + 1/a$ and $B = -b/a$, in which case we have $u = (\gamma - 1)/(1 - Bg)$ and $g\varphi = (1 - Bg)/(\gamma - 1)$. From the conditions (3.3) and the general property of the solutions for arbitrary $\varphi(g)$ /1/, consisting of the fact that the density behind the shock wave cannot be everywhere greater than the initial density, it follows that only two domains of the parameters γ and B need be considered, namely either $\gamma > 1$ or $(1 - \gamma)/2 < B < 1$, or $\gamma < -1$ and $1 < B < (1 - \gamma)/2$. Let

$$\Phi = \begin{cases}\dfrac{\gamma + 1}{(\gamma - 1)(1 - B)}(2Y)^{3/(\gamma-2)}(1 - Y)^{1/(2\gamma-1)}\left[\dfrac{5\gamma - 4}{\gamma + 1}Y + \dfrac{2 - \gamma}{\gamma + 1}\right]^{\Gamma}, & |\gamma| > 1,\ \gamma \neq 2 \\ \dfrac{3}{1 - B}(2Y)^{-1/2}(1 - Y)^{1/3}\exp[1/(3Y) - 1], & \gamma = 2\end{cases} \tag{3.5}$$

$(\Gamma = (4 - 5\gamma)/[(\gamma - 2)(2\gamma - 1)])$.

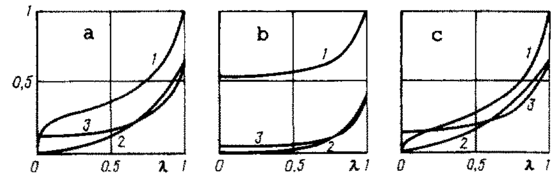

Fig.1

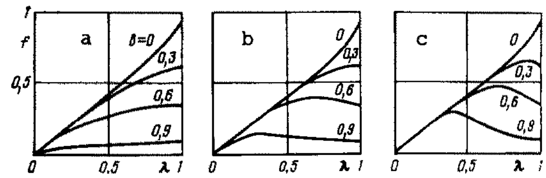

Fig.2

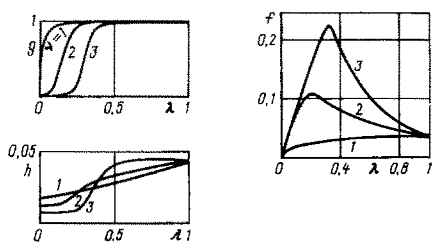

Fig.3

The solution of Problem (2.6), (2.9) of the form

$$g = \Phi\,(1 + B\Phi)^{-1} \tag{3.6}$$

together with (2.10) yields, for $Y \in [1/3; 1)$, an exact parametric solution of the problem of an intense explosion for $|\gamma| > 1$ and $\nu = 1$, continuable up to the plane of symmetry. The asymptotic behaviour of the solution near the plane of symmetry is identical, for $\gamma > 1$, with the well-known behaviour of the solution for a real gas at $\nu = 1$ (the parameter Y can be eliminated) $g \sim \lambda^{1/(\gamma-1)},\ h \sim 1,\ f \sim \lambda$ for $\lambda \ll 1$, and for $\gamma < -1$ $g \sim 1,\ h \sim 1,\ f \sim \lambda^{(\gamma-1)/\gamma}$, and the density g is equal to $1/B$ when $\lambda = 0$.

Notes. 1°. The internal energy of a disperse two-phase system in which one phase in incompressible and the other is a real gas, and the phases are in equilibrium with regard to velocities and temperatures, is given by the function $g\varphi = (1 - Bg)/(\gamma - 1)$ where $\gamma > 1$ is the effective adiabatic index (less than that of the gas), and B is a constant equal, for $g_1 = 1$, to the volume fraction of the incompressible phase before the shock wave $(0 < B < 1)$. Examples of such media, which we shall call simply disperse media, are a dusty gas, a mixture of gas and liquid droplets, and a liquid containing bubbles.

An attempt was made in /3/ to solve the problem of an intense explosion in such a disperse medium (at $0 < B < 1$) using a variable transformation and starting from the known solution /2/ for a real gas $(B = 0)$. The transformations proposed in /3/ were found to be excact only in the case of $\nu = 1$.

A parametric solution for disperse media was given above for $\nu = 1$, as well as three other solutions, $(\gamma > 1,\ (1-\gamma)/2 < B < 0;\ \gamma < -1$ and $1 < B < (1-\gamma)/2;\ \varphi \equiv b$ for $0 < b < 1/2)$ for the case when the parameters γ and B have no specific physical meaning.

2°. The results obtained enable us to obtain an exact solution of the problem of an intense explosion at a plane boundary dividing two different ideal compressible media for each of which the exact solution is known. The distribution of the energy of the explosion between two half-spaces is found from the condition that the (dimensional) pressures are equal to each other at the boundary separating the two media. Only one exact solution was known earlier, namely the solution for the problem where the explosion occurs at the boundary between two real gases (such problems are discussed in greater detail in /8/, Chapter 5).

4. Figs.1-3 show the exact solutions obtained above, as well as some results obtained numerically using the Runge-Kutta method for Problem (1.2)-(1.4) for all values of ν, for the functions $\varphi = b$ and $g\varphi = (1 - Bg)/(\gamma - 1)$.

Fig.1 shows the exact solutions for $\nu = 1$ for the non-disperse media, in the form of graphs showing the dependence of g/g_s (curve 1), f (2), and h (3) on λ at $b = 0.3$ (a); $\gamma = -2$, $B = 1.2$ (b); $\gamma = 5$ and $B = -1$ (c). Numerical computations of the same versions at $\nu = 2;\ 3$ have shown only an insignificant deformation of the graphs, but in the case of version b at $\nu = 3$, a cavity of non-zero density was found ($h(\lambda) = 0$ for $\lambda \leqslant 0.17$).

A model of disperse media was studied by numerical methods before the publication of /3/, for $\nu = 1$ /5/ and $\nu = 2$ /6/. Therefore we shall give below only those results of the numerical analysis which were not mentioned in the above papers.

Fig.2 shows, for $\gamma = 1.3$, graphs of relations $f(\lambda)$ for $\nu = 1$(a), $\nu = 2$ (b), $\nu = 3$ (c), for various values of B (0; 0.3; 0.6; 0.9). We see the non-monotonic form of the function $f(\lambda)$ at $\nu = 2;\ 3$ for $B > B_*$ ($B_* \approx 0.2$ at $\nu = 3$; $B_* \approx 0.3$ for $\nu = 2$), and in the case of spherical symmetry this property is more pronounced. The non-monotonic form of the function $f(\lambda)$ has not been detected in /6/, since in that paper $B \leqslant 0.04$.

When the values of B are close to unity, the disperse medium behaves as an incompressible medium. This effect is shown for $\gamma = 1.3$ and $B = 0.95$ in Fig.3, where graphs are given showing the dependence of g, f, h on λ for $\nu = 1;\ 2;\ 3$. We note that in the cylindrical and spherical cases, segments of nearly constant density can be formed only when the velocity is sufficiently strongly non-monotonic. It can be shown /1/ that when $\nu = 3$, the limit solution ($B = 1$) for disperse media will be the well-known solution /2/ for an incompressible fluid with a cavity expanding from the centre of the explosion.

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